Math 300: Midterm 2

INSTRUCTOR: DR. PALMER, SPRING 2019

Name & ID: Solutions!

April 17, 2019
100 points possible
80 minutes time limit
Remember to make your proofs and arguments as clear as possible
Show your work!

Notation:

$\varnothing$	empty set
$\mathbb{R}$	set of real numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{Z}$	set of integers
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}_m$	the integers modulo m
$\mathrm{Dom}(R)$	domain of the relation R
$\mathrm{Rng}(R)$	range of the relation R
$f\colon A \to B$	function from A to B
surjective	onto
injective	one-to-one

Good luck!

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(12)	(10)	(14)	(10)	(16)	(12)	(14)	(14)	(100)

Problem 1. Computations. (12 points - 4 points each)

(a) What is the lowest positive integer equivalent to 4^{62} modulo 7? (Your answer should be an integer between 0 and 6, inclusive).

Notice

$4^2 = 16 = 2 \pmod 7$

$4^3 = 2 \cdot 4 = 1 \pmod 7$. $\rightarrow$ $4^{60} = (1)^{20} = 1 \pmod 7$

$\rightarrow$ $4^{62} = (4)^{60} \cdot 4^2 = 2 \pmod 7$

$\boxed{2}$

(b) What is the remainder when $(802) \cdot (1679)$ is divided by 8?

$802 = 800 + 2 \Rightarrow 802 = 2 \pmod 8$

$1679 = 1680 - 1 \Rightarrow 1679 = 7 \pmod 8$

Thus,

$(802)(1679) = 2(7) = 14 = 6 \pmod 8$

$\boxed{6}$

(c) Let $x \in \mathbb{Z}$ with $0 \leq x \leq 11$ and $5x = 2 \mod 12$. Find x.

Notice $5 \cdot 5 = 25 = 1 \pmod{12}$

So $5x = 2 \pmod{12} \Rightarrow 5(5x) = 5(2) \mod 12$

$\Rightarrow 25x = 10 \mod 12$

$\Rightarrow x = 10 \mod 12$.

$\boxed{x = 10}$

Problem 2. State whether the following is **true** or **false**, and then **prove your answer**. (10 points - 5 points each)

(a) Let $x, y \in \mathbb{Z}$. **Prove** or $\boxed{\textbf{disprove}}$: $5x = 5y \mod 10$ implies $x = y \mod 10$.

FALSE.

Pf: Let $x = 1$ and $y = 3$.

Then $5x = 5(1) = 5 \pmod{10}$

and $5y = 5(3) = 15 = 5 \pmod{10}$

so $5x = 5y \pmod{10}$

but $x \neq y \pmod{10}$ because $1 \neq 3 \pmod{10}$. ☒

(b) Let $x, y \in \mathbb{Z}$. $\boxed{\textbf{Prove}}$ or **disprove**: $5x = 5y \mod 9$ implies $x = y \mod 9$.

TRUE

Pf: Suppose $x, y \in \mathbb{Z}$ with $5x = 5y \pmod 9$.

Then $2(5x) = 2(5y) \pmod 9$

$\Rightarrow 10x = 10y \pmod 9$

$\Rightarrow x = y \pmod 9$

since $10 = 1 \pmod 9$. ☒

Problem 3. Let $A = \{1, 2, 3, 4\}$ and $B = \{100, 101, 102\}$. Let $R, S \subset A \times B$ be given by $R = \{(1, 100), (1, 101), (2, 102), (3, 102), (4, (100)\}$ and $S = \{(1, 102), (2, 101), (3, 100)\}$. (14 points)

(a) Find $\text{Dom}(R)$ and $\text{Dom}(S)$. (3 points)

$\text{Dom}(R) = \{1, 2, 3, 4\}$

$\text{Dom}(S) = \{1, 2, 3\}$

(b) Find $\text{Rng}(R)$ and $\text{Rng}(S)$. (3 points)

$\text{Rng}(R) = \{100, 101, 102\}$

$\text{Rng}(S) = \{100, 101, 102\}$

(c) **Prove** or **disprove**: R is a function from A to B. (4 points)

FALSE. Pf: Since $(1,100), (1,101) \in R$ we see R is not well-defined. ☒

(d) **Prove** or **disprove**: S is a function from A to B. (4 points)

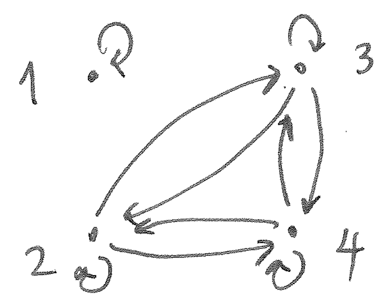
FALSE.

Pf: From part (a) notice $\text{Dom}(S) \neq A$ so S is not a function from A to B. ☒

Problem 4. Answer the following. (10 points - 5 points each)

(a) Let $A = \{1, 2, 3, 4\}$ and $\mathcal{P} = \{\{1\}, \{2, 3, 4\}\}$. Find the relation R on A associated to the partition $\mathcal{P}$. (Give R as a set of ordered pairs)

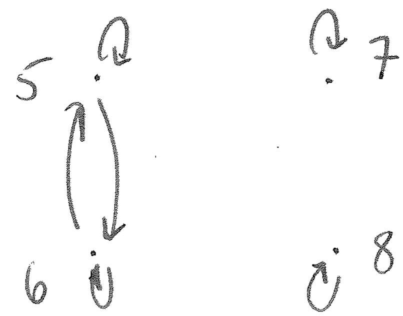
$$R = \{(1,1), (2,2), (3,3), (4,4), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3)\}$$



(6,5)

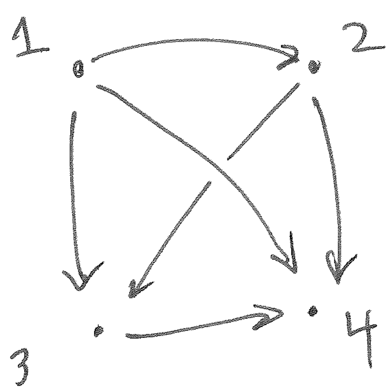
(b) Let $B = \{5, 6, 7, 8\}$ and let $S \subset B \times B$ be given by $S = \{(5, 6), (5, 5), (6, 6), (7, 7), (8, 8)\}$. The relation S is an equivalence relation (you don't have to show this). Find the partition associated to S.

$$\mathcal{P} = \{\{5, 6\}, \{7\}, \{8\}\}$$

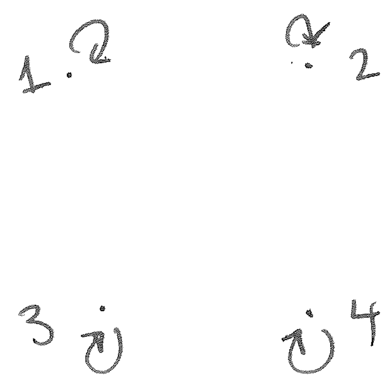


Problem 5. For each relation drawn as a graph below, answer the following "yes" or "no": (16 points - 4 points for each graph)

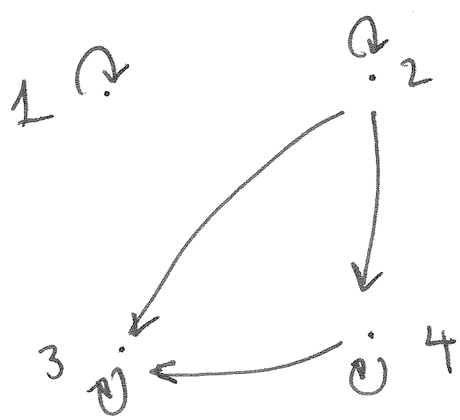
(a) Is is reflexive? **(b)** Is it symmetric? **(c)** Is it transitive? **(d)** Is it an equivalence relation?



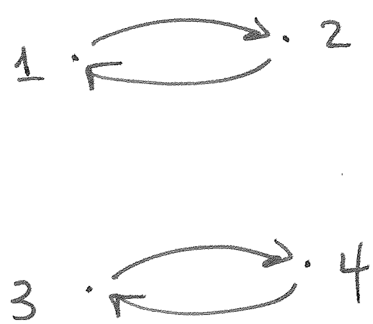
(a) no
(b) no
(c) yes
(d) no



(a) yes
(b) yes
(c) yes
(d) yes



(a) yes
(b) no
(c) yes
(d) no

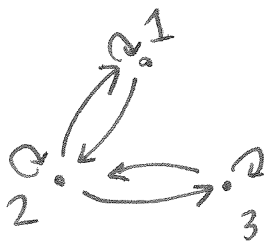


(a) no
(b) yes
(c) no
(d) no

Problem 6. Let $X = \{1, 2, 3\}$. Give an example of each of the following types of relations on X, write them as a list of ordered pairs or draw a directed graph. (12 points - 3 points each)

(a) Given an example of a relation R on X which is symmetric and reflexive but not an equivalence relation.

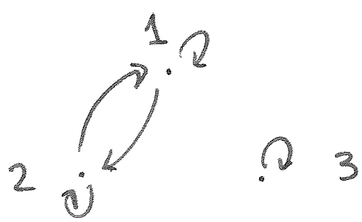
$R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2)\}$



(b) Given an example of an equivalence relation S on X.

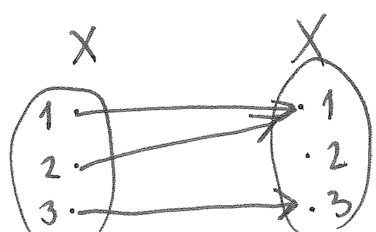
$S = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

(There are many answers!)



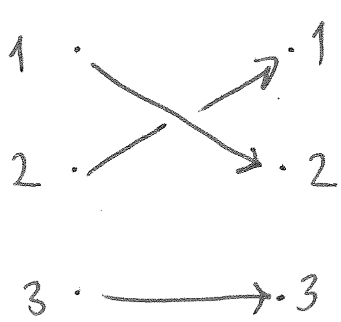
(c) Given an example of a function $f : X \to X$ which is **not** a bijection.

$f = \{(1,1), (2,1), (3,3)\}$



(d) Given an example of a bijection $g : X \to X$ which is not equal to the identity function, I_X.

$g = \{(1,2), (2,1), (3,3)\}$



Problem 7. Let A, B be sets. Let $f\colon A \to B$ be a function from A to B. (14 points)

(a) By definition, what does it mean to say that the function g is the inverse of f? (That is, I am asking you to state the definition of an *inverse function*) (6 points)

(1) g is a function $g: B \to A$

(2) $g \circ f = I_A$

(3) $f \circ g = I_B$

$f: A \to B$

(b) Suppose that f is an invertable function with inverse given by the function g. Show that f is a bijection. (**Hint:** There should be two parts to the proof: showing that f is one-to-one and showing that f is onto) (8 points)

Pf: Suppose $f: A \to B$ is an invertable function with inverse g.

To show f injective, let $x, y \in A$ with $f(x) = f(y)$. Applying g to both sides gives

$$g \circ f(x) = g \circ f(y).$$

Since $g \circ f = I_A$ this implies $x = y$.

To show f surjective let $y \in B$ be arbitrary. Define $t \in A$ by $g(y)$. Then

$$f(t) = f \circ g(y) = y$$

as desired, since $f \circ g = I_B$. ∎

Problem 8. Prove the following. (14 points)

(a) Let X, Y, Z be sets. Let $u\colon X \to Y$ and $v\colon Y \to Z$ both be onto. Prove that $v \circ u\colon X \to Z$ is also onto. (6 points)

Pf: Let u, v be as in the statement.

Pick arbitrary $y \in Z$.

Since $v: Y \to Z$ is onto $\exists$ $b \in Y$ with $v(b) = y$.

Since $w: X \to Y$ is onto $\exists$ $a \in X$ with $u(a) = b$.

Then $a \in X$ and

$$v \circ u(a) = v(u(a)) = v(b) = y.$$

$\blacksquare$

(b) Let A, B, C be sets. Let $f\colon A \to B$ and $g\colon B \to C$ be functions such that f is onto the set B and $g \circ f\colon A \to C$ is one-to-one. Prove that g is one-to-one. (8 points)

Pf: Let f, g be as in the statement.

Let $x, y \in B$ with $g(x) = g(y)$. and we will show $x = y$.

Since f is onto $\exists$ $a, b \in A$ with $f(a) = x$, $f(b) = y$.

Thus

$$g(x) = g(y) \Rightarrow g(f(a)) = g(f(b)) \Rightarrow g \circ f(a) = g \circ f(b)$$

$$\Rightarrow a = b$$

since $g \circ f$ is one-to-one. Since $a = b$ we see $f(a) = f(b)$, but $x = f(a)$, $y = f(b)$ so we conclude $x = y$, as desired. $\blacksquare$